

The Jiang–Su algebra revisited

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Abstract

We give a number of new characterizations of the Jiang–Su algebra \mathcal{Z} , both intrinsic and extrinsic, in terms of C^* -algebraic, dynamical, topological and K -theoretic conditions. Along the way we study divisibility properties of C^* -algebras, we give a precise characterization of those unital C^* -algebras of stable rank one that admit a unital embedding of the dimension-drop C^* -algebra $Z_{n,n+1}$, and we prove a cancellation theorem for the Cuntz semigroup of C^* -algebras of stable rank one.

1 Introduction

In Elliott’s program to classify nuclear C^* -algebras by K -theory data (see [14] for an introduction), the systematic use of strongly self-absorbing C^* -algebras play a central role. The term “strongly self-absorbing C^* -algebras” was formally coined in the paper [18] to denote the class of C^* -algebras $\mathcal{D} \neq \mathbb{C}$ for which there is an isomorphism from \mathcal{D} to $\mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the embedding $d \mapsto d \otimes 1$. Strongly self-absorbing C^* -algebras are automatically simple, nuclear and have at most one tracial state. The Cuntz algebras \mathcal{O}_2 and \mathcal{O}_∞ and the Jiang–Su algebra \mathcal{Z} are strongly self-absorbing.

Most classification results obtained so far can be interpreted as *classification up to \mathcal{D} -stability*, where \mathcal{D} is one of the (few) known strongly self-absorbing examples (cf. [16]). The classification of Kirchberg algebras can thus be viewed as classification up to \mathcal{O}_∞ -stability. There is at present much interest in classification up to \mathcal{Z} -stability, which appears to be the largest possible class of “ \mathcal{D} -stable” C^* -algebras. One may view \mathcal{Z} as being the stably finite analogue of \mathcal{O}_∞ .

The original construction of the Jiang–Su algebra in [8] is as an inductive limit of a sequence of C^* -algebras with specified connecting mappings. Whereas everything in this construction in principle is concrete, the presentation is not canonical, and it depends on infinitely many choices. Since the Jiang–Su algebra has become to play such a central role in the classification program it is desirable to have a more concrete and “finite” presentation of this algebra, or to be able to characterize it in a more streamlined way. We refer to the recent paper by Dadarlat and Toms, [3], for a very nice such characterization. In this paper we present other characterizations and presentations of the Jiang–Su algebra.

The many alternative descriptions available for the Cuntz algebra \mathcal{O}_∞ provide a guideline of what kind of characterizations one might expect for \mathcal{Z} . They involve (C^* -)algebraic, dynamical and K -theoretic conditions; in the present paper we shall employ similar conditions to characterize the Jiang–Su algebra in various manners. We also give a topological characterization of \mathcal{Z} which currently has no known analogue for \mathcal{O}_∞ .

Besides its original presentation as a universal C^* -algebra with generators and relations, \mathcal{O}_∞ may be written as a crossed product of a canonical subalgebra by an endomorphism (see [2]). Both these descriptions are concrete, and entirely intrinsic. Kirchberg has obtained a completely different characterization of \mathcal{O}_∞ , as the uniquely determined purely infinite, separable, unital,

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nuclear C^* -algebra which is KK -equivalent to the complex numbers (cf. [9]; see also [10]). Note that this description is not intrinsic, since it compares \mathcal{O}_∞ with the complex numbers (at least on the level of KK -theory). Using the well known facts that strongly self-absorbing C^* -algebras are nuclear and either stably finite with a unique tracial state or purely infinite, it is then immediate that \mathcal{O}_∞ is the unique strongly self-absorbing C^* -algebra that has no tracial state and is KK -equivalent to \mathbb{C} . Moreover, one might rephrase the condition of being *purely* infinite in terms of the Cuntz semigroup: a simple C^* -algebra is purely infinite if and only if it is infinite and has almost unperforated Cuntz semigroup, in which case its Cuntz semigroup coincides with the semigroup $\{0, \infty\}$. Regarding the Cuntz semigroup as a K -theoretic invariant in the broadest sense, one arrives at an abstract (but extrinsic) characterization of \mathcal{O}_∞ among strongly self-absorbing C^* -algebras in terms of K -theory data.

Let us compare the characterizations of \mathcal{O}_∞ and of \mathcal{Z} in more detail. Cuntz's original description of \mathcal{O}_∞ uses (infinitely many) generators and relations. While Jiang and Su's construction is not quite of this type, the building blocks of their inductive limit are given by (finitely many) generators and relations—and for many purposes this has proven to be just as useful as if the whole algebra was presented as a universal C^* -algebra.

Cuntz's description of \mathcal{O}_∞ as a crossed product uses the dynamics of a certain canonical subalgebra. It is not so easy to write the Jiang–Su algebra as a crossed product by a single endomorphism, since such algebras tend to have nontrivial K_1 -groups, but we can nonetheless use dynamical properties of certain canonical subalgebras to write \mathcal{Z} as a stationary inductive limit of such a subalgebra; the connecting map is not easy to describe explicitly (its existence follows from a result of I. Hirshberg and the authors), but its pertinent property can be stated in a very elegant manner. More precisely, we show that the Jiang–Su algebra is a stationary inductive limit of a generalized prime dimension drop C^* -algebra and a trace-collapsing endomorphism; any such limit is isomorphic to \mathcal{Z} . Although the connecting maps of the inductive system are not given explicitly, this is still an entirely intrinsic description of the Jiang–Su algebra. We wish to point out that this picture has already proven highly useful in [22].

The largest part of the paper will be devoted to finite versions (for \mathcal{Z}) of Kirchberg's characterization of \mathcal{O}_∞ . The general pattern of such characterizations goes as follows: one states various conditions (C^* -algebraic, K -theoretic and/or topological), and shows that, if met by a strongly self-absorbing C^* -algebra \mathcal{D} , then \mathcal{D} is isomorphic to \mathcal{Z} . Then one observes that \mathcal{Z} itself satisfies the conditions in question. The latter will follow mostly from known results by Jiang and Su, the first named author, and the second named author and E. Kirchberg. To establish an isomorphism between \mathcal{D} and \mathcal{Z} , it will suffice to construct unital embeddings in both directions, as we work within the class of strongly self-absorbing C^* -algebras.

Our first characterization singles out \mathcal{Z} as the uniquely determined strongly self-absorbing C^* -algebra of stable rank one, for which the unit can be approximately divided in the Cuntz semigroup, and which is absorbed by any UHF algebra. The latter condition will guarantee that the algebra in question is absorbed by the Jiang–Su algebra, using a joint result by I. Hirshberg and the authors. That the algebra absorbs \mathcal{Z} follows from stable rank one together with a cancellation theorem for the Cuntz semigroup established in Section 4, and from the divisibility condition. The key technical tool here will be Proposition 5.1, which provides criteria for embeddability of certain dimension drop intervals into a unital C^* -algebra. Essentially, this is done by analyzing a set of generators and relations quite different from those used to describe dimension drop intervals in [8].

Along similar lines, we then obtain another characterization of \mathcal{Z} , as the uniquely determined strongly self-absorbing finite C^* -algebra which has almost unperforated Cuntz semigroup and which is absorbed by any UHF algebra. We point out that the latter condition in particular entails that the algebra in question is KK -equivalent to the complex numbers, whence this characterization indeed may be viewed as a finite analogue of Kirchberg's characterization of \mathcal{O}_∞ . Again, the proof uses ideas from Proposition 5.1 in a crucial way, along with a further careful analysis of divisibility properties of strongly self-absorbing C^* -algebras.

Our last characterization of the Jiang–Su algebra involves the decomposition rank, a notion of covering dimension for nuclear C^* -algebras introduced by E. Kirchberg and the second named

author in [11]. Our result says that \mathcal{Z} is the uniquely determined strongly self-absorbing C^* -algebra with finite decomposition rank which is KK -equivalent to the complex numbers. The proof uses the fact that finite decomposition rank entails sufficient regularity on the level of the Cuntz semigroup; together with Proposition 5.1 this shows that finite decomposition rank and strongly self-absorbing imply \mathcal{Z} -stability. That \mathcal{Z} is the only such algebra then follows from a recent classification theorem of the second named author ([22]). We note that decomposition rank is of a very topological flavour, and that there is currently no analogous characterization for \mathcal{O}_∞ .

The paper is organized as follows. In Section 2, we recall some background results about strongly self-absorbing C^* -algebras, the Jiang–Su algebra, and order zero maps. In Section 3, we characterize the Jiang–Su algebra as a stationary inductive limit of generalized dimension drop algebras. Section 4 provides a cancellation theorem for the Cuntz semigroup of C^* -algebras with stable rank one. In Section 5 we derive an abstract characterization of the Jiang–Su algebra among strongly self-absorbing C^* -algebras of stable rank one; in the subsequent section we obtain a variation of this result, asking the Cuntz semigroup to be almost unperforated. Finally, in Section 7, we characterize the Jiang–Su algebra among strongly self-absorbing C^* -algebras of finite decomposition rank.

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2 Some background results

In this section we recall some well-known results about strongly self-absorbing C^* -algebras in general and about the Jiang–Su algebra, \mathcal{Z} , in particular. (The reader is referred to the introduction and to [18] for a definition and properties of strongly self-absorbing C^* -algebras.) We also recall some facts about completely positive contractive (c.p.c.) order zero maps.

We quote below a result by Andrew Toms and the second named author about the hierarchy of strongly self-absorbing C^* -algebras:

Proposition 2.1 (Toms–Winter, [18]) *Let \mathcal{D} and \mathcal{E} be strongly self-absorbing C^* -algebras. Then:*

- (i) *\mathcal{D} embeds unitaly into \mathcal{E} if and only if $\mathcal{D} \otimes \mathcal{E}$ is isomorphic to \mathcal{E} .*
- (ii) *\mathcal{D} and \mathcal{E} are isomorphic if \mathcal{D} embeds unitaly into \mathcal{E} and \mathcal{E} embeds unitaly into \mathcal{D} .*

For each supernatural number p let M_p denote the UHF algebra of type p . We say that p is of *infinite type* if $p^\infty = p$, in which case M_p is strongly self-absorbing. (If p is a natural number, then M_p will denote the C^* -algebra of $p \times p$ matrices over the complex numbers.) If p and q are natural or supernatural numbers, then we set

$$Z_{p,q} = \{f \in C([0, 1], M_p \otimes M_q) \mid f(0) \in M_p \otimes \mathbb{C}, f(1) \in \mathbb{C} \otimes M_q\}.$$

If p and q are natural numbers, then $Z_{p,q}$ is a so-called *dimension-drop C^* -algebra*. If p and q are relatively prime, then $Z_{p,q}$ is said to be *prime*.

It is worthwhile noting that $Z_{p,q}$ has no non-trivial projections (other than 0 and 1) if and only if p and q are relatively prime (natural or supernatural numbers), and that its K -theory in that case is given by

$$K_0(Z_{p,q}) \cong \mathbb{Z}, \quad K_1(Z_{p,q}) = 0.$$

Prime dimension-drop C^* -algebras play a crucial role in the definition of the Jiang–Su algebra:

Theorem 2.2 (Jiang–Su, [8]) *The inductive limit of the sequence*

$$A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \cdots,$$

where each A_j is a prime dimension-drop C^ -algebra and where the connecting mappings are unital, is isomorphic to the Jiang–Su algebra \mathcal{Z} if and only if it is simple and has a unique tracial state.*

Let p be a natural number. Recall from [20] that a c.p.c. map $\varphi: M_p \rightarrow A$ is said to have *order zero* if it preserves orthogonality. We collect below some well known facts about order zero maps (see [20, Proposition 3.2(a)] and [21, 1.2] for Proposition 2.3, and [19, 1.2.3] for Proposition 2.4). We let e_{ij} , or sometimes $e_{ij}^{(p)}$, denote the canonical (i, j) th matrix unit in M_p .

Proposition 2.3 (Winter, [20, 21]) *Let A be a C^* -algebra, let $p \in \mathbb{N}$, and let $\varphi: M_p \rightarrow A$ be a c.p.c. order zero map.*

- (i) *There is a unique $*$ -homomorphism $\tilde{\varphi}: C_0([0, 1]) \otimes M_p \rightarrow A$ such that $\varphi(x) = \tilde{\varphi}(\iota \otimes x)$ for all $x \in M_p$, where $\iota(t) = t$.*
- (ii) *There is a unique $*$ -homomorphism $\bar{\varphi}: M_p \rightarrow A^{**}$ given by sending the matrix unit e_{ij} in M_p to the partial isometry in A^{**} in the polar decomposition of $\varphi(e_{ij})$. We have*

$$\varphi(x) = \bar{\varphi}(x)\varphi(1_p) = \varphi(1_p)\bar{\varphi}(x)$$

for all $x \in M_p$; and $\bar{\varphi}(1_p)$ is the support projection of $\varphi(1_p)$.

- (iii) *If, for some $h \in A^{**}$ with $\|h\| \leq 1$, the element h^*h commutes with $\bar{\varphi}(M_p)$ and satisfies $h^*h\bar{\varphi}(M_p) \subseteq A$, then the map $\varphi_h: M_p \rightarrow A$ given by $\varphi_h(x) = h\bar{\varphi}(x)h^*$, for $x \in M_p$, is a well defined c.p.c. order zero map.*

The map $\bar{\varphi}$ in (ii) above will be called the *supporting $*$ -homomorphism* of φ .

Proposition 2.4 (Winter, [19]) *Suppose $x_1, x_2, \dots, x_p \in A$ satisfy the relations*

$$\|x_i\| \leq 1, \quad x_1 \geq 0, \quad x_i x_i^* = x_1^* x_1, \quad x_j^* x_j \perp x_i^* x_i, \quad (\mathcal{R}_p)$$

for all $i, j = 1, \dots, n$ with $i \neq j$. Then the linear map $\psi: M_p \rightarrow A$ given by $\psi(e_{ij}) = x_i^ x_j$ is a c.p.c. order zero map.*

Note that the original version of the above result was phrased in terms of elements of the form e_{i1} , $i = 2, \dots, p$. However, it is straightforward to check that the two versions are in fact equivalent.

The next proposition contains a recipe for finding a unital $*$ -homomorphism from a dimension drop C^* -algebra $Z_{p,q}$ into a unital C^* -algebra A .

Proposition 2.5 *Let A be a unital C^* -algebra. For relatively prime natural numbers p and q , suppose that $\alpha: M_p \rightarrow A$ and $\beta: M_q \rightarrow A$ are c.p.c. order zero maps satisfying*

$$\alpha(1_p) + \beta(1_q) = 1_A, \quad [\alpha(M_p), \beta(M_q)] = 0. \quad (2.1)$$

Then there is a (unique) unital $$ -homomorphism $\varphi: Z_{p,q} \rightarrow A$, which makes the diagram*

$$\begin{array}{ccc} & Z_{p,q} & \\ \nearrow & \downarrow \varphi & \nwarrow \\ C_0([0, 1], M_p) & & C_0((0, 1], M_q) \\ \searrow \tilde{\alpha}' & & \swarrow \tilde{\beta} \\ & A & \end{array}$$

commutative, where the upwards maps are the obvious ones, where $\tilde{\alpha}$ and $\tilde{\beta}$ are as in Proposition 2.3(i), and where $\tilde{\alpha}'$ is obtained from $\tilde{\alpha}$ by reversing the orientation of the interval $[0, 1]$.

Proof: By [8, Proposition 7.3], $Z_{p,q}$ is the universal C^* -algebra with generators $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q$ and relations (\mathcal{R}_p) from Proposition 2.4 (with the x_i 's replaced by the a_i 's), (\mathcal{R}_q) (with the x_i 's replaced by the b_i 's), and

$$[a_i, b_j] = 0, \quad [a_i, b_j^*] = 0, \quad \sum_{k=1}^p a_k^* a_k + \sum_{l=1}^q b_l^* b_l = 1,$$

for $i = 1, \dots, p$ and $j = 1, \dots, q$. Identifying $Z_{p,q}$ with a sub- C^* -algebra of $C([0, 1]) \otimes M_p \otimes M_q$ in the canonical way, and letting $\iota \in C([0, 1])$ denote the function $\iota(t) = t$, we can take the generators in $Z_{p,q}$ to be

$$a_i = (1 - \iota)^{1/2} \otimes e_{1i}^{(p)} \otimes 1_q, \quad b_j = \iota^{1/2} \otimes 1_p \otimes e_{1j}^{(q)}.$$

It is straightforward to check that the elements

$$\bar{a}_i = \alpha(1_p)^{1/2} \bar{\alpha}(e_{1i}^{(p)}) = \bar{\alpha}'((1 - \iota)^{1/2} \otimes e_{1i}^{(p)}), \quad \bar{b}_j = \beta(1_q)^{1/2} \bar{\beta}(e_{1j}^{(q)}) = \bar{\beta}(\iota^{1/2} \otimes e_{1j}^{(q)})$$

in A satisfy the relations above, where $\bar{\alpha}$ and $\bar{\beta}$ are the supporting $*$ -homomorphisms for α and β , respectively. By the universal property of $Z_{p,q}$ there is (precisely) one unital $*$ -homomorphism $\varphi: Z_{p,q} \rightarrow A$ such that $\varphi(a_i) = \bar{a}_i$ and $\varphi(b_j) = \bar{b}_j$ for all i and j ; and one checks (on elements of the form $(1 - \iota)^{1/2} \otimes e_{1i}^{(p)} \in C_0([0, 1], M_p)$ and $\iota^{1/2} \otimes e_{1j}^{(q)} \in C_0([0, 1], M_q)$) that the diagram in the proposition is commutative. \square

3 The Jiang–Su algebra and the C^* -algebras $Z_{p,q}$

In this section we characterize the Jiang–Su algebra using dynamical properties of the C^* -algebras $Z_{p,q}$ (defined in the previous section, and with p and q supernatural numbers). The first result is an immediate consequence of one of the main result from [7]:

Proposition 3.1 *Let \mathcal{D} be a strongly self-absorbing C^* -algebra which tensorially is absorbed by every UHF-algebra B , i.e., $\mathcal{D} \otimes B \cong B$. Then $\mathcal{D} \otimes Z_{p,q} \cong Z_{p,q}$ whenever p and q are infinite supernatural numbers.*

Proof: The C^* -algebra $Z_{p,q}$ is in a canonical way a $C([0, 1])$ -algebra with fibres being UHF-algebras of type p at the left end-point, of type pq at $(0, 1)$, and of type q at the right end-point. Each fibre is accordingly a UHF-algebra and so absorbs \mathcal{D} tensorially. As the interval $[0, 1]$ has finite dimension it follows from [7] that $Z_{p,q}$ also absorbs \mathcal{D} . \square

The Jiang–Su algebra is strongly self-absorbing ([18]) and it is being absorbed by all UHF-algebras ([8]), and so we get:

Corollary 3.2 *Let p and q be infinite supernatural numbers. Then $Z_{p,q}$ absorbs the Jiang–Su algebra: $\mathcal{Z} \otimes Z_{p,q} \cong Z_{p,q}$.*

The proposition below is proved in [15, Proposition 2.2] in the case where $p = n^\infty$ and $q = m^\infty$, and where n and m are natural numbers, that are relatively prime. We shall need this result in the slightly more general case where p and q are arbitrary supernatural numbers that are relatively prime. Assume that such p and q are given. Then write M_p and M_q as inductive limits

$$M_{p_1} \rightarrow M_{p_2} \rightarrow M_{p_3} \rightarrow \dots \rightarrow M_p, \quad M_{q_1} \rightarrow M_{q_2} \rightarrow M_{q_3} \rightarrow \dots \rightarrow M_q$$

(with unital connecting mappings) for suitable sequences of natural numbers $\{p_j\}$ and $\{q_j\}$. As $p_j | p$ and $q_j | q$, it is automatic that p_j and q_j are relatively prime for all j . Let $\sigma_j: M_{p_j} \otimes M_{q_j} \rightarrow M_{p_{j+1}} \otimes M_{q_{j+1}}$ be a unital $*$ -homomorphism such that $\sigma_j(M_{p_j} \otimes \mathbb{C}) \subseteq M_{p_{j+1}} \otimes \mathbb{C}$ and $\sigma_j(\mathbb{C} \otimes M_{q_j}) \subseteq \mathbb{C} \otimes M_{q_{j+1}}$. Then $Z_{p,q}$ is the limit of the inductive system

$$Z_{p_1, q_1} \xrightarrow{\rho_1} Z_{p_2, q_2} \xrightarrow{\rho_2} Z_{p_3, q_3} \xrightarrow{\rho_3} \dots \longrightarrow Z_{p, q},$$

where ρ_j is given by $\rho_j(f) = \sigma_j \circ f$. Proceeding as in the proof of [15, Proposition 2.2] one obtains the following:

Proposition 3.3 *Let p and q be supernatural numbers that are relatively prime. Then $Z_{p,q}$ embeds unitaly into \mathcal{Z} .*

Combining Proposition 3.3 and Corollary 3.2 we get unital embeddings $Z_{p,q} \rightarrow \mathcal{Z} \rightarrow Z_{p,q}$ whenever p and q are infinite supernatural numbers that are relatively prime. As we shall see below, this characterizes \mathcal{Z} among strongly self-absorbing C^* -algebras. First we note a related result.

A unital endomorphism φ on a unital C^* -algebra A is said to be *trace-collapsing* if $\tau \circ \varphi = \tau' \circ \varphi$ for any pair of tracial states τ and τ' on A .

Theorem 3.4 *Let p and q be infinite supernatural numbers that are relatively prime.*

- (i) *There exists a trace-collapsing unital endomorphism on $Z_{p,q}$.*
- (ii) *Let φ be any trace-collapsing unital endomorphism on $Z_{p,q}$. Then the Jiang–Su algebra \mathcal{Z} is isomorphic to the inductive limit of the stationary inductive sequence:*

$$Z_{p,q} \xrightarrow{\varphi} Z_{p,q} \xrightarrow{\varphi} Z_{p,q} \xrightarrow{\varphi} \cdots$$

Proof: (i). Take the composition of any unital embeddings $Z_{p,q} \rightarrow \mathcal{Z} \rightarrow Z_{p,q}$ (cf. the remarks above) and recall (eg. from [8]) that \mathcal{Z} has a unique trace.

(ii). We note first that the inductive limit, call it A , of the sequence above is an inductive limit of prime dimension-drop C^* -algebras, i.e., of C^* -algebras of the form $Z_{n,m}$ with n and m natural numbers that are relatively prime. Indeed, each $Z_{p,q}$ is such an inductive limit, cf. the remarks above. Hence A can locally be approximated by prime dimension-drop C^* -algebras. Each (prime) dimension-drop C^* -algebra is weakly stable by [8, Proposition 7.3], whence any C^* -algebra that locally can be approximated by prime dimension-drop C^* -algebras is an actual inductive limit of them, cf. [12].

It now follows from Jiang and Su, [8], cf. Theorem 2.2, that A is isomorphic to the Jiang–Su algebra \mathcal{Z} if and only if A is simple and has unique trace.

Uniqueness of the trace of A follows easily from the assumption that φ is trace-collapsing.

The endomorphism φ is necessarily injective. Indeed, if it were not and I is the kernel of φ , then φ would induce an embedding of $Z_{p,q}/I$ into $Z_{p,q}$. But any non-trivial quotient of $Z_{p,q}$ has non-trivial projections (i.e., projections other than 0 and 1), whereas $Z_{p,q}$ only contains the trivial projections, cf. the remarks in Section 2.

That A is simple now follows from the fact that $\varphi(a)$ is full in $Z_{p,q}$ for all non-zero $a \in Z_{p,q}$. To see this, let $\pi_t : Z_{p,q} \rightarrow M_{pq}$ denote the fibre map (for $t \in [0, 1]$). Let τ be the (unique) tracial state on M_{pq} . Then $t \mapsto (\tau \circ \pi_t \circ \varphi)(a^*a)$ is constant by the assumption that φ is trace-collapsing, and this function is non-zero (because a is non-zero and φ is injective). Hence $\pi_t(\varphi(a)) \neq 0$ for all $t \in [0, 1]$, which entails that $\varphi(a)$ is full in $Z_{p,q}$. \square

Proposition 3.5 *The Jiang–Su algebra \mathcal{Z} is the only strongly self-absorbing C^* -algebra for which there are relatively prime infinite supernatural numbers p and q and unital embeddings $Z_{p,q} \rightarrow \mathcal{Z} \rightarrow Z_{p,q}$.*

Proof: Suppose that p and q are infinite supernatural numbers that are relatively prime and that A is a strongly self-absorbing C^* -algebra for which there are unital $*$ -homomorphisms $\lambda : Z_{p,q} \rightarrow A$ and $\mu : A \rightarrow Z_{p,q}$. Consider the inductive system

$$A \xrightarrow{\mu} Z_{p,q} \xrightarrow{\lambda} A \xrightarrow{\mu} Z_{p,q} \xrightarrow{\lambda} A \xrightarrow{\mu} Z_{p,q} \xrightarrow{\lambda} \cdots$$

The inductive limit of this system coincides with the inductive limits of the two subsystems below:

$$A \xrightarrow{\lambda \circ \mu} A \xrightarrow{\lambda \circ \mu} A \xrightarrow{\lambda \circ \mu} \cdots, \quad Z_{p,q} \xrightarrow{\mu \circ \lambda} Z_{p,q} \xrightarrow{\mu \circ \lambda} Z_{p,q} \xrightarrow{\mu \circ \lambda} \cdots.$$

Any unital endomorphism on a strongly self-absorbing C^* -algebra is approximately unitarily equivalent to the identity by [18, Corollary 1.12]. It thus follows from an inductive limit argument (after Elliott — see for example [14, Corollary 2.3.3]) that the former inductive system above has inductive limit isomorphic to A .

As A has unique trace (cf. [18, Theorem 1.7]) the unital endomorphism $\mu \circ \lambda$ is trace-collapsing. Hence the latter of the two inductive systems above has limit isomorphic to \mathcal{Z} by Theorem 3.4.

This proves that A is isomorphic to \mathcal{Z} . \square

4 A cancellation theorem for the Cuntz semigroup

In this section we prove a cancellation theorem for the Cuntz semigroup for C^* -algebras of stable rank one. This result, which might be of independent interest, and which extends a recent result of Elliott, [5], is needed for the next section.

We refer the reader to [15] and [13] for notation and background material on Cuntz comparison of positive elements and on the Cuntz semigroup.

Recall the following fact, proved in [13]:

Proposition 4.1 *Let A be a unital C^* -algebra of stable rank one, let a, b be positive elements in A such that $a \precsim b$, and let $\varepsilon > 0$. It follows that there is a unitary element $u \in A$ such that*

$$u^*(a - \varepsilon)_+ u \in \overline{bAb}.$$

The two results below show that the Cuntz semigroup $W(A)$ of a C^* -algebra of stable rank one has almost cancellation:

Proposition 4.2 *Let A be a C^* -algebra of stable rank one, let a, b be positive elements in $M_\infty(A)$, and let p be a projection in $M_\infty(A)$ such that*

$$a \oplus p \precsim b \oplus p.$$

Then $a \precsim b$.

Proof: Upon replacing A by a suitable matrix algebra over A we can assume that a, b, p all belong to A and that $a \perp p$ and $b \perp p$. Let $0 < \varepsilon < 1$. As $(p - \varepsilon)_+ = (1 - \varepsilon)p$, we can use Proposition 4.1 to find a unitary u in the unitization of A such that

$$u((a - \varepsilon)_+ + p)u^* \in \overline{(b + p)A(b + p)} \stackrel{\text{def}}{=} B.$$

Being a hereditary sub- C^* -algebra of A , B and hence also its unitization are of stable rank one. Now, upu^* and p are equivalent projections in B , and so there is a unitary v in the unitization of B (that we may regard as being a sub- C^* -algebra of the unitization of A) such that $upu^* = vpv^*$. Note that

$$v^*u(a - \varepsilon)_+u^*v \in B, \quad v^*u(a - \varepsilon)_+u^*v \perp v^*upu^*v = p,$$

which entails that $v^*u(a - \varepsilon)_+u^*v$ belongs to $(1 - p)B(1 - p) = \overline{bAb}$. This proves that $(a - \varepsilon)_+ \precsim b$; and as $\varepsilon > 0$ was arbitrary, we conclude that $a \precsim b$. \square

Theorem 4.3 (Cancellation) *Let A be a C^* -algebra of stable rank one, and let x, y be elements in the Cuntz semigroup $W(A)$ such that*

$$x + \langle c \rangle \leq y + \langle (c - \varepsilon)_+ \rangle.$$

for some $c \in M_\infty(A)^+$ and for some $\varepsilon > 0$. Then $x \leq y$.

Proof: Upon replacing A by a matrix algebra over A we can assume that c belongs to A , and that $x = \langle a \rangle$, $y = \langle b \rangle$ for some positive elements a, b in A with $a \perp c$ and $b \perp c$. Next, upon adjoining a unit to A we may assume that A is unital (this will not affect the comparison of the elements a, b, c). Let $h_\varepsilon: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be given by

$$h_\varepsilon(t) = \begin{cases} \varepsilon^{-1}(\varepsilon - t), & 0 \leq t \leq \varepsilon, \\ 0, & t \geq \varepsilon. \end{cases} \quad (4.1)$$

Then $(c - \varepsilon)_+ \perp h_\varepsilon(c)$ and $c + h_\varepsilon(c)$ is invertible. Hence

$$\begin{aligned} a \oplus 1_A &\precsim a \oplus (c + h_\varepsilon(c)) \precsim a \oplus c \oplus h_\varepsilon(c) \\ &\precsim b \oplus (c - \varepsilon)_+ \oplus h_\varepsilon(c) \precsim b \oplus ((c - \varepsilon)_+ + h_\varepsilon(c)) \\ &\precsim b \oplus 1_A. \end{aligned}$$

The claim now follows from Proposition 4.2. \square

One cannot strengthen Theorem 4.3 to the more intuitive statement: $x + z \leq y + z$ implies $x \leq y$, when x, y, z are elements in the Cuntz semigroup, $W(A)$, of an arbitrary C^* -algebra A of stable rank one. Indeed, if one takes A to be a UHF algebra with trace τ , p to be a projection, and a, b to be positive elements in A such that

$$\tau(p) = d_\tau(a) (= \lim_{n \rightarrow \infty} \tau(a^{1/n})),$$

and such that 0 is an accumulation point of $\text{sp}(a) \setminus \{0\}$ and of $\text{sp}(b) \setminus \{0\}$, then $p \not\precsim a$ but $p \oplus b \precsim a \oplus b$ (see [1] for more details).

We shall also need the lemma below for the next section. First we fix some notation to be used here and in the sequel.

Notation 4.4 For positive numbers $0 \leq \eta < \varepsilon \leq 1$ define continuous functions $f_\varepsilon, g_{\eta, \varepsilon}: [0, 1] \rightarrow \mathbb{R}^+$ by

$$g_{\eta, \varepsilon}(t) = \begin{cases} 0, & t \leq \eta, \\ 1, & \varepsilon \leq t \leq 1, \\ \text{linear}, & \text{else,} \end{cases} \quad f_\varepsilon = g_{0, \varepsilon}.$$

Lemma 4.5 *Let A be a unital C^* -algebra of stable rank one, and let $a, b \in A^+$ be such that $\langle a \rangle + \langle b \rangle \geq \langle 1_A \rangle$. Then $1_A - f_\varepsilon(a) \precsim (b - \varepsilon)_+$ for some $\varepsilon > 0$.*

Proof: As $\langle (1_A - \varepsilon)_+ \rangle = \langle 1_A \rangle$ for all $\varepsilon \in [0, 1]$ one can conclude from [13] that here exists $\delta > 0$ such that

$$\langle (a - \delta)_+ \rangle + \langle (b - \delta)_+ \rangle \geq \langle 1_A \rangle.$$

Take ε such that $0 < \varepsilon < \delta$. Observe that $1_A - f_\varepsilon(a) \perp (a - \varepsilon)_+$. It follows that

$$\begin{aligned} \langle 1_A - f_\varepsilon(a) \rangle + \langle (a - \varepsilon)_+ \rangle &\leq \langle 1_A \rangle \leq \langle (a - \delta)_+ \rangle + \langle (b - \delta)_+ \rangle \leq \langle (b - \varepsilon)_+ \rangle + \langle (a - \delta)_+ \rangle \\ &= \langle (b - \varepsilon)_+ \rangle + \langle ((a - \varepsilon)_+ - (\delta - \varepsilon))_+ \rangle. \end{aligned}$$

By Theorem 4.3 this implies that $1_A - f_\varepsilon(a) \precsim (b - \varepsilon)_+$. \square

5 An axiomatic description of the Jiang–Su algebra

The main result of this section is Theorem 5.5 below in which a new characterization of the Jiang–Su algebra is given. The proof uses facts about the Cuntz semigroup and comparison theory for positive elements derived in the previous section.

Two positive elements a and b in a C^* -algebra A are said to be *equivalent*, written $a \sim b$, if there is $x \in A$ such that $a = x^*x$ and $b = xx^*$. It is easy to see that $a \sim b$ implies $\langle a \rangle = \langle b \rangle$ in $W(A)$. Recall the definition of the dimension-drop C^* -algebra $Z_{p, q}$ from Section 2.

Proposition 5.1 *Let A be a unital C^* -algebra of stable rank one, and let n be a natural number. The following four conditions are equivalent:*

- (i) *There exists $x \in W(A)$ such that $nx \leq \langle 1_A \rangle \leq (n+1)x$.*
- (ii) *There exist $\varepsilon > 0$ and mutually equivalent and orthogonal positive elements b_1, b_2, \dots, b_n in A such that*

$$1_A - (b_1 + b_2 + \dots + b_n) \precsim (b_1 - \varepsilon)_+.$$

- (iii) *There are elements $v, s_1, s_2, \dots, s_n \in A$ of norm 1 such that*

$$s_1^* s_1 = s_i^* s_i^*, \quad s_i^* s_i \perp s_j^* s_j, \quad v^* v = 1_A - \sum_{k=1}^n s_k^* s_k, \quad vv^* s_1^* s_1 = vv^*, \quad (5.1)$$

for all i and j with $i \neq j$.

- (iv) *There is a unital $*$ -homomorphism from the C^* -algebra $Z_{n,n+1}$ into A .*

The hypothesis of stable rank one is only needed for the implication (i) \Rightarrow (ii).

Proof: (i) \Rightarrow (ii). Find a positive element d in some matrix algebra $M_k(A)$ over A such that $x = \langle d \rangle$. There is $\delta > 0$ such that $(n+1)\langle (d-\delta)_+ \rangle \geq \langle 1_A \rangle$ (cf. [13]). As $n\langle d \rangle \leq \langle 1_A \rangle$ there is a row matrix $t \in M_{1,nk}(A)$ such that

$$t^* t = t^* 1_A t = (d-\delta)_+ \oplus (d-\delta)_+ \oplus \dots \oplus (d-\delta)_+.$$

Write $t = (t_1, t_2, \dots, t_n)$ with $t_i \in M_{1,k}(A)$. Then

$$t_i^* t_j = \begin{cases} (d-\delta)_+, & i = j, \\ 0, & i \neq j. \end{cases}$$

Put $e_j = t_j t_j^*$. Then e_1, e_2, \dots, e_n are pairwise orthogonal positive elements in A each of which is equivalent to $(d-\delta)_+$. It follows in particular that

$$\langle 1_A \rangle \leq (n+1)\langle (d-\delta)_+ \rangle = (n+1)\langle e_1 \rangle = \langle e_1 + e_2 + \dots + e_n \rangle + \langle e_1 \rangle.$$

Now use Lemma 4.5 (and recall from 4.4 the definition of the function f_η) to see that there exists $\eta > 0$ such that

$$1_A - f_\eta(e_1 + e_2 + \dots + e_n) \precsim (e_1 - \eta)_+.$$

Put $b_j = f_\eta(e_j)$. Note that $e_1 \precsim f_\eta(e_1) = b_1$ (and also $b_1 \precsim e_1$), so there exists $\varepsilon > 0$ such that $(e_1 - \eta)_+ \precsim (b_1 - \varepsilon)_+$ (see [13]). It now follows that the elements b_1, b_2, \dots, b_n are as desired, because $f_\eta(e_1 + e_2 + \dots + e_n) = b_1 + b_2 + \dots + b_n$.

(ii) \Rightarrow (iii). We may assume that $\varepsilon < 1$. Since each b_i is equivalent to b_1 , there are $x_2, \dots, x_n \in A$ such that $x_i x_i^* = b_1$ and $x_i^* x_i = b_i$. Let $x_i = v_i |x_i| = |x_i^*| v_i$ be the polar decomposition with v_i a partial isometry in A^{**} . Put $s_1 = f_\varepsilon(b_1)^{1/2}$ and put $s_i = v_i f_\varepsilon(b_i)^{1/2} = f_\varepsilon(b_1)^{1/2} v_i \in A$ for $i = 2, 3, \dots, n$ (cf. 4.4). Then $s_i s_i^* = f_\varepsilon(b_1) = s_1^* s_1$ for all i , and $s_i^* s_i = f_\varepsilon(b_i)$ whence $s_i^* s_i \perp s_j^* s_j$ when $i \neq j$.

Note that $1 - (f_\varepsilon(b_1) + \dots + f_\varepsilon(b_n))$ belongs to the hereditary sub- C^* -algebra generated by $(1 - (b_1 + \dots + b_n) - (1 - \varepsilon))_+$. Choose $0 < \eta < 1 - \varepsilon$ and note that $g_{\eta, 1-\varepsilon}(1 - (b_1 + \dots + b_n))$ is a unit for $(1 - (b_1 + \dots + b_n) - (1 - \varepsilon))_+$ and hence also for $1 - (f_\varepsilon(b_1) + \dots + f_\varepsilon(b_n))$ (cf. 4.4). It follows from the hypothesis and [13, Proposition 2.4] that there is $x \in A$ such that

$$x^* x = g_{\eta, 1-\varepsilon}(1 - (b_1 + \dots + b_n)), \quad x x^* \in \overline{(b_1 - \varepsilon)_+ A (b_1 - \varepsilon)_+}.$$

Set

$$v = x(1_A - (f_\varepsilon(b_1) + \dots + f_\varepsilon(b_n)))^{1/2}.$$

Then

$$v^*v = 1_A - (f_\varepsilon(b_1) + \cdots + f_\varepsilon(b_n)) = 1_A - \sum_{k=1}^n s_k^* s_k.$$

Since vv^* belongs to $\overline{(b_1 - \varepsilon)_+ A (b_1 - \varepsilon)_+}$ and $(b_1 - \varepsilon)_+ f_\varepsilon(b_1) = (b_1 - \varepsilon)_+$, we get that

$$vv^* s_1^* s_1 = vv^* f_\varepsilon(b_1) = vv^*.$$

(iii) \Rightarrow (iv). In the light of Proposition 2.5, it suffices to construct order zero c.p.c. maps $\alpha: M_{n+1} \rightarrow A$ and $\beta: M_n \rightarrow A$ with commuting images such that $\alpha(1_{n+1}) + \beta(1_n) = 1_A$.

The construction of α and β (and the verification that they have the desired properties) is rather long and tedious. It may be constructive to note that one quite easily can write down order zero c.p.c. maps $\mu: M_{n+1} \rightarrow A$ and $\rho: M_n \rightarrow A$ such that $\mu(1_{n+1}) + \rho(1_n) \geq 1_A$. Indeed, put

$$t_1 = (v^*v)^{1/2}, \quad t_{j+1} = v^* s_j \quad (j = 1, 2, \dots, n).$$

One can easily verify that the elements s_1, s_2, \dots, s_n satisfy the relations (\mathcal{R}_n) of Proposition 2.4, and that t_1, t_2, \dots, t_{n+1} satisfy the relations (\mathcal{R}_{n+1}) . It therefore follows from Proposition 2.4 that there are order zero c.p.c. maps

$$\mu: M_{n+1} \rightarrow A, \quad \mu(e_{ij}^{(n+1)}) = t_i^* t_j, \quad \rho: M_n \rightarrow A, \quad \rho(e_{ij}^{(n)}) = s_i^* s_j.$$

These maps fail to have commuting images, and $\mu(1_{n+1}) + \rho(1_n)$ is larger than but not equal to 1_A . We shall in the following modify these maps so that they get the desired properties. In the process we shall make much use of the map ρ , but we shall make no further explicit use of the map μ .

Upon replacing s_1 by $(s_1^* s_1)^{1/2}$ we may assume that $s_1 \geq 0$. Let $\bar{v}|v|$ be the polar decomposition of v with \bar{v} a partial isometry in A^{**} and $|v| = (v^*v)^{1/2}$. Let us note some relations satisfied by the elements $v, \bar{v}, s_1, \dots, s_n$ to be used later in the proof:

- (a) $s_1 v = v, s_1 \bar{v} = \bar{v}$.
- (b) $s_j v = s_j \bar{v} = 0$ for $j = 2, 3, \dots, n$.
- (c) $vv^* \perp v^*v$,
- (d) $s_i s_j = 0$ for all $i = 2, 3, \dots, n, j = 1, 2, \dots, n$.
- (e) $[s_i, v^*v] = [s_i^*, v^*v] = 0$ for all $i = 1, \dots, n$,
- (f) $c \bar{v}^* \bar{v} = c$ for all $c \in \overline{v^* v A v^* v}$.
- (g) $\bar{v} c \in A$ for all $c \in \overline{v^* v A v^* v}$.

The first part of (a) follows by the hypothesis that $vv^* = s_1^* s_1 vv^*$, and the second part of (a) follows from the first part and standard properties of the polar decomposition. To see (b) use that $s_j^* s_j vv^* = s_j^* s_j s_1^* s_1 vv^* = 0$. Next,

$$v^* v \cdot vv^* = (1_A - \sum_{j=1}^n s_j^* s_j) vv^* \stackrel{(b)}{=} (1_A - s_1^* s_1) vv^* \stackrel{(5.1)}{=} 0,$$

whence (c) holds. For $i \neq 1$ we have $s_i^* s_i s_j s_j^* = s_i^* s_i s_1^* s_1 = 0$, so (d) holds. For $i = 1, 2, \dots, n$ one has

$$v^* v s_i = (1_A - \sum_{j=1}^n s_j^* s_j) s_i \stackrel{(d)}{=} s_i - s_1^* s_1 s_i \stackrel{(5.1)}{=} s_i - s_i s_i^* s_i \stackrel{(d), (5.1)}{=} s_i (1_A - \sum_{j=1}^n s_j^* s_j) = s_i v^* v.$$

This proves (e). (f) and (g) are well-known properties of the polar decomposition.

Recall the definition of the order zero c.p.c. map ρ from above, and associate to it the supporting *-homomorphism $\bar{\rho}: M_n \rightarrow A^{**}$ defined in Proposition 2.3 (ii). Note (from (5.1) and Proposition 2.3 (ii)) that

(h) $\rho(1_n) = 1_A - v^*v$,

(i) $\rho(1_n)\bar{\rho}(x) = \rho(x) \in A$ for all $x \in M_n$.

Define a map $\varphi: \overline{v^*vAv^*v} \rightarrow A$ by

$$\varphi(c) = \sum_{i=1}^n s_i^* \bar{v} c \bar{v}^* s_i, \quad c \in \overline{v^*vAv^*v},$$

cf. (g). The map φ is clearly linear and hermitian, and, as shown below, it is actually a $*$ -homomorphism. Take $c_1, c_2 \in \overline{v^*vAv^*v}$ and calculate:

$$\begin{aligned} \varphi(c_1)\varphi(c_2) &= \sum_{i,j=1}^n s_i^* \bar{v} c_1 \bar{v}^* s_i s_j^* \bar{v} c_2 \bar{v}^* s_j = \sum_{i=1}^n s_i^* \bar{v} c_1 \bar{v}^* s_i s_i^* \bar{v} c_2 \bar{v}^* s_i \\ &= \sum_{i=1}^n s_i^* \bar{v} c_1 \bar{v}^* s_i^* s_i \bar{v} c_2 \bar{v}^* s_i \stackrel{(a)}{=} \sum_{i=1}^n s_i^* \bar{v} c_1 \bar{v}^* \bar{v} c_2 \bar{v}^* s_i \\ &\stackrel{(f)}{=} \sum_{i=1}^n s_i^* \bar{v} c_1 c_2 \bar{v}^* s_i = \varphi(c_1 c_2), \end{aligned}$$

where we in the second and third equation have used the relations for the s_i 's from (5.1). We note the following relations concerning the $*$ -homomorphism φ :

(j) $\varphi(c)\bar{v} = \bar{v}c$ for all $c \in \overline{v^*vAv^*v}$,

(k) $[\varphi(c), s_i] = [\varphi(c), s_i^*] = 0$ for all $c \in \overline{v^*vAv^*v}$ and $i = 1, 2, \dots, n$,

(l) $[\varphi(c), \bar{\rho}(x)] = 0$ for all $c \in \overline{v^*vAv^*v}$ and $x \in M_n$,

(m) $\varphi(\overline{v^*vAv^*v}) \perp \overline{v^*vAv^*v}$.

We first prove (j):

$$\varphi(c)\bar{v} \stackrel{(b)}{=} s_1^* \bar{v} c \bar{v}^* s_1 \bar{v} \stackrel{(a)}{=} \bar{v} c \bar{v}^* \bar{v} \stackrel{(f)}{=} \bar{v} c.$$

Next, for $i = 1, 2, \dots, n$ we have

$$\varphi(c)s_i \stackrel{(d)}{=} s_1^* \bar{v} c \bar{v}^* s_1 s_i \stackrel{(a)}{=} \bar{v} c \bar{v}^* s_i \stackrel{(a)}{=} s_1^* s_1 \bar{v} c \bar{v}^* s_i \stackrel{(5.1)}{=} s_i s_i^* \bar{v} c \bar{v}^* s_i \stackrel{(5.1)}{=} s_i \varphi(c),$$

hence (k) holds. The image of $\bar{\rho}$ is contained in the weak closure (in A^{**}) of the C^* -algebra generated by the s_i 's, so (l) follows from (k). The calculation:

$$\varphi(v^*v)v^*v = \sum_{j=1}^n s_j^* \bar{v} v^* v \bar{v}^* s_j v^*v = \sum_{j=1}^n s_j^* v v^* s_j v^*v \stackrel{(e)}{=} \sum_{j=1}^n s_j^* v v^* v^* v s_j \stackrel{(c)}{=} 0,$$

shows that (m) holds.

Consider the two C^* -algebras:

$$\begin{aligned} D_1 &= \{f \in C_0([0, 1], M_n) \mid f(0) \in \mathbb{C} \cdot 1_n\}, \\ D_2 &= \{f \in C_0([0, 1], M_n \otimes M_n) \mid f(0) \in \mathbb{C} \cdot 1_n \otimes 1_n\}. \end{aligned}$$

Note (by (e)) that v^*v commutes with $\rho(M_n)$ and (hence) with $\bar{\rho}(M_n)$. The $*$ -homomorphism $\bar{\lambda}: C([0, 1], M_n) = C([0, 1]) \otimes M_n \rightarrow A^{**}$ given by

$$\bar{\lambda}(f \otimes x) = f(1 - v^*v)\bar{\rho}(x), \quad f \in C([0, 1]), \quad x \in M_n,$$

restricts to a $*$ -homomorphism $\lambda: D_1 \rightarrow \overline{v^*vAv^*v}$. Indeed, D_1 is generated as a C^* -algebra by the elements $(1 - \iota) \otimes 1_n$ and $\iota(1 - \iota) \otimes x$, $x \in M_n$, (where $\iota(t) = t$), and

$$\bar{\lambda}((1 - \iota) \otimes 1_n) = v^*v, \quad \bar{\lambda}(\iota(1 - \iota) \otimes x) = (1 - v^*v)v^*v\bar{\rho}(x) \stackrel{(h),(i)}{=} v^*v\rho(x) \in \overline{v^*vAv^*v}.$$

By (l) we can define a $*$ -homomorphism $\gamma: D_2 \rightarrow A$ by

$$\gamma(f \otimes x \otimes y) = (\varphi \circ \lambda)(f \otimes x)\bar{\rho}(y), \quad f \in C_0([0, 1]), \quad x, y \in M_n.$$

To see that the image of γ is contained in A (rather than in A^{**}) observe that the image of φ is contained in the hereditary sub- C^* -algebra of A generated by $\sum_{i=1}^n s_i^*s_i = 1_A - v^*v = \rho(1_n)$, and so by (i), $\varphi(c)\bar{\rho}(y)$ belongs to A for all $c \in \overline{v^*vAv^*v}$ and $y \in M_n$.

Let $u \in C([0, 1], M_n \otimes M_n) \cap \mathcal{M}(D_2)$ be a self-adjoint unitary such that

$$u(t) = 1_n \otimes 1_n \quad (0 \leq t \leq 1/3), \quad u(t)(x \otimes y)u(t) = y \otimes x \quad (2/3 \leq t \leq 1),$$

for all $x, y \in M_n$. Put $w = \gamma^{**}(u) \in A^{**}$ (where $\gamma^{**}: D_2^{**} \rightarrow A^{**}$ is the canonical extension of γ). Let $g \in C_0([0, 1])$ be given by

$$g(t) = \begin{cases} 1, & 0 \leq t \leq 2/3, \\ 0, & t = 1, \\ \text{linear}, & 2/3 < t < 1. \end{cases}$$

We list some easily verified identities involving u , w , and g .

- (n) $u \cdot (g \otimes 1_n \otimes 1_n) \in D_2$,
- (o) $w \cdot (\varphi \circ \lambda)(g \otimes 1_n) = (\varphi \circ \lambda)(g \otimes 1_n) \cdot w = \gamma(u \cdot (g \otimes 1_n \otimes 1_n)) \in A$.
- (p) $((1 - g) \otimes 1_n \otimes 1_n) \cdot u \cdot (1 \otimes x \otimes y) \cdot u = (1 - g) \otimes y \otimes x$ for all $x, y \in M_n$.

Put

$$x_i = \lambda(g \otimes 1_n)^{1/2} \bar{v}^* s_i w \quad (i = 1, 2, \dots, n), \quad x_{n+1} = \lambda(g \otimes 1_n)^{1/2}.$$

The x_i 's satisfy the following relations:

- (q) $x_i^* x_{n+1} = w(\varphi \circ \lambda)(g \otimes 1_n) \rho(e_{i1}^{(n)}) \bar{v}$ for $i = 1, 2, \dots, n$.
- (r) $x_i^* x_j \in A$ for all $i, j = 1, 2, \dots, n+1$.
- (s) $x_j x_j^* = \lambda(g \otimes 1_n) = x_{n+1}^* x_{n+1}$ for $j = 1, 2, \dots, n+1$.
- (t) $x_i^* x_i \perp x_j^* x_j$ for $i \neq j$.
- (u) $\sum_{i=1}^n s_i^* \bar{v} \bar{v}^* s_i \varphi(c) = \varphi(c)$ for all $c \in \overline{v^*vAv^*v}$.
- (v) $\sum_{j=1}^{n+1} x_j^* x_j = \lambda(g \otimes 1_n) + (\varphi \circ \lambda)(g \otimes 1_n)$.

Let us verify these identities. For $i = 1, 2, \dots, n$ we have $x_i^* x_{n+1} = w s_i^* \bar{v} \lambda(g \otimes 1_n)$ (recall that $w = w^*$) and

$$\begin{aligned} s_i^* \bar{v} \lambda(g \otimes 1_n) &\stackrel{(j)}{=} s_i^* (\varphi \circ \lambda)(g \otimes 1_n) \bar{v} \stackrel{(k)}{=} (\varphi \circ \lambda)(g \otimes 1_n) s_i^* \bar{v} \\ &\stackrel{(a)}{=} (\varphi \circ \lambda)(g \otimes 1_n) s_i^* s_1 \bar{v} = (\varphi \circ \lambda)(g \otimes 1_n) \rho(e_{i1}^{(n)}) \bar{v}. \end{aligned}$$

This proves that (q) holds. As $s_i^* \bar{v} \lambda(g \otimes 1_n)$ belongs to A (by (g)), we can use (o) to conclude that $x_i^* x_{n+1}$ belongs to A . When $i, j = 1, 2, \dots, n$ we have $x_i^* x_j = w^* s_i^* \bar{v} \lambda(g \otimes 1_n) \bar{v}^* s_j w$; and $\bar{v} \lambda(g \otimes 1_n) \bar{v}^*$ belongs to A by (g). Moreover,

$$s_i^* \bar{v} \lambda(g \otimes 1_n) \bar{v}^* s_j \stackrel{(j),(k)}{=} (\varphi \circ \lambda)(g \otimes 1_n)^{1/2} s_i^* \bar{v} \bar{v}^* s_j (\varphi \circ \lambda)(g \otimes 1_n)^{1/2}.$$

We can now use (o) to see that $x_i^* x_j$ belongs to A . To see that (s) holds, note that $\bar{v}^* s_j w w^* s_j^* \bar{v} = \bar{v}^* s_1^* s_1 \bar{v} = \bar{v}^* \bar{v}$ and use (f). Use (5.1) to see that $x_i^* x_i \perp x_j^* x_j$ when $i \neq j$ and $i, j \leq n$. We proceed to establish (u):

$$\sum_{i=1}^n s_i^* \bar{v} \bar{v}^* s_i \varphi(c) \stackrel{(5.1)}{=} \sum_{i=1}^n s_i^* \bar{v} \bar{v}^* s_i s_i^* \bar{v} c \bar{v}^* s_i \stackrel{(5.1)}{=} \sum_{i=1}^n s_i^* \bar{v} \bar{v}^* s_1^* s_1 \bar{v} c \bar{v}^* s_i \stackrel{(a)}{=} \sum_{i=1}^n s_i^* \bar{v} c \bar{v}^* s_i = \varphi(c).$$

Next,

$$\begin{aligned} \sum_{i=1}^n x_i^* x_i &= \sum_{i=1}^n w^* s_i^* \bar{v} \lambda(g \otimes 1_n) \bar{v}^* s_i w \stackrel{(j),(k)}{=} \sum_{i=1}^n w^* (\varphi \circ \lambda)(g \otimes 1_n) s_i^* \bar{v} \bar{v}^* s_i w \\ &\stackrel{(u)}{=} w^* (\varphi \circ \lambda)(g \otimes 1_n) w \stackrel{(o)}{=} (\varphi \circ \lambda)(g \otimes 1_n). \end{aligned}$$

From this we see that (v) holds, and we also see that $x_i^* x_i \perp x_{n+1}^* x_{n+1}$ (cf. (m)).

It follows from (r), (s), (t) and Proposition 2.4 that there is an order zero c.p.c. map

$$\alpha: M_{n+1} \rightarrow A,$$

given by

$$\alpha(e_{ij}^{(n+1)}) = x_i^* x_j \quad (i, j = 1, 2, \dots, n+1).$$

We list some properties of α :

- (w) $\alpha(1_{n+1}) = \lambda(g \otimes 1_n) + (\varphi \circ \lambda)(g \otimes 1_n)$
- (x) $[\lambda(g \otimes 1_n), \bar{\rho}(M_n)] = 0, \quad [\alpha(1_{n+1}), \bar{\rho}(M_n)] = 0.$
- (y) $1_A - \alpha(1_{n+1}) \in \overline{\rho(1_n) A \rho(1_n)}.$
- (z) $(1_A - \alpha(1_{n+1})) \bar{\rho}(M_n) \subseteq A.$

(w) is just a reformulation of (v). The first part of (x) follows from (e) when we note that g is a function of $1 - \iota$, whence $\lambda(g \otimes 1_n)$ belongs to the C^* -algebra generated by $\lambda((1 - \iota) \otimes 1_n) = v^* v$, and that the image of $\bar{\rho}$ is contained in the weak closure (in A^{**}) of the C^* -algebra generated by the s_i 's. The second part of (x) follows from the first part together with (w) and (l). As $g \geq 1 - \iota$ we get $\lambda(g \otimes 1_n) \geq \lambda((1 - \iota) \otimes 1_n) = v^* v$, whence $1_A - \alpha(1_{n+1}) \leq 1_A - v^* v = \rho(1_n)$. This proves (y). Finally, (z) follows from (y) and the fact that $\rho(1_n) \bar{\rho}(x) = \rho(x) \in A$.

It follows from (x) and (z) above, together with Proposition 2.3 (iii), that

$$\beta(x) = (1_A - \alpha(1_{n+1})) \bar{\rho}(x), \quad x \in M_n,$$

defines an order zero c.p.c. map from M_n into A . Use (y) to see that $(1_A - \alpha(1_{n+1})) \bar{\rho}(1_n) = 1_A - \alpha(1_{n+1})$, whence $\alpha(1_{n+1}) + \beta(1_n) = 1_A$. To complete the proof we must show that the images of α and β commute. For brevity, put $h = \lambda(g \otimes 1_n)$, recall that $\alpha(1_{n+1}) = h + \varphi(h)$ and

that $h \perp \varphi(h)$ (the latter by (m)). For $k, l, i = 1, 2, \dots, n$ we have:

$$\begin{aligned}
\beta(e_{kl}^{(n)})\alpha(e_{i,n+1}^{(n+1)}) &\stackrel{(q)}{=} (1 - h - \varphi(h))\bar{\rho}(e_{kl}^{(n)})w\varphi(h)\rho(e_{i1}^{(n)})\bar{v} \\
&\stackrel{(i),(l),(o)}{=} (1 - \varphi(h))\varphi(h)\bar{\rho}(e_{kl}^{(n)})w\bar{\rho}(e_{i1}^{(n)})\rho(1_n)\bar{v} \\
&= \gamma\left((1 - g)g \otimes 1_n \otimes 1_n\right)(1 \otimes 1_n \otimes e_{kl}^{(n)})u(1 \otimes 1_n \otimes e_{i1}^{(n)})\rho(1_n)\bar{v} \\
&\stackrel{(p)}{=} \gamma\left((1 - g)g \otimes 1_n \otimes 1_n\right)u(1 \otimes 1_n \otimes e_{i1}^{(n)})(1 \otimes e_{kl}^{(n)} \otimes 1_n)\rho(1_n)\bar{v} \\
&= (1 - \varphi(h))\varphi(h)w\bar{\rho}(e_{i1}^{(n)})(\varphi \circ \lambda)(1 \otimes e_{kl}^{(n)})\rho(1_n)\bar{v} \\
&\stackrel{(k)}{=} \varphi(h)(1 - \varphi(h))w\bar{\rho}(e_{i1}^{(n)})\rho(1_n)(\varphi \circ \lambda)(1 \otimes e_{kl}^{(n)})\bar{v} \\
&\stackrel{(i),(j)}{=} \varphi(h)(1 - \varphi(h))w\rho(e_{i1}^{(n)})\bar{v}\lambda(1 \otimes e_{kl}^{(n)}) \\
&\stackrel{(o),(k),(j)}{=} \varphi(h)w\rho(e_{i1}^{(n)})\bar{v}(1 - h)\bar{\rho}(e_{kl}^{(n)}) \\
&\stackrel{(m)}{=} \varphi(h)w\rho(e_{i1}^{(n)})\bar{v}(1 - h - \varphi(h))\bar{\rho}(e_{kl}^{(n)}) \\
&= \alpha(e_{i,n+1}^{(n+1)})\beta(e_{kl}^{(n)})
\end{aligned}$$

The image of α is contained in the C^* -algebra, E , generated by $\{\alpha(e_{i,n+1}) \mid i = 1, 2, \dots, n\}$, which again, by the argument above, is contained in the commutant of the image of β . (To see this use that $\alpha(1_{n+1}) \in E$, that E is contained in the hereditary sub- C^* -algebra of A generated by $\alpha(1_{n+1})$, and that $\alpha(x)\alpha(y) = \alpha(1_{n+1})\alpha(xy)$ for all $x, y \in M_{n+1}$, cf. Proposition 2.3 (ii).) It has now been verified that the images of α and β commute.

(iv) \Rightarrow (i). This follows from [15, Lemma 4.2]. \square

Remark 5.2 Any stably finite unital \mathcal{Z} -stable C^* -algebra satisfies conditions (i) through (iv) of Proposition 5.1.

Quite surprisingly there is a very recent example of a unital, simple infinite dimensional C^* -algebra that does not admit a unital embedding of the Jiang–Su algebra or for that matter of any dimension drop C^* -algebra $Z_{n,m}$ with $n, m \geq 2$ (see [4]). This example is based on Example 4.8 of [7] of a unital $C(X)$ -algebra whose fibres absorb the Jiang–Su algebra, but which does not itself absorb the Jiang–Su algebra. This C^* -algebra has no finite dimensional quotient, and one can quite easily see that one cannot unittally embed the Jiang–Su algebra or $Z_{n,m}$ (with $n, m \geq 2$) into this C^* -algebra.

In other words, simple infinite dimensional C^* -algebras can fail to have the (very weak) divisibility property 5.1 (i). Nonetheless, prompted by the equivalence of (i) and (iv) of the proposition above, one might ask the following:

Question 5.3 Does the Jiang–Su algebra \mathcal{Z} embed unittally into any unital C^* -algebra A for which its Cuntz semigroup $W(A)$ has the following divisibility property: For every natural number n there exists $x \in W(A)$ such that $nx \leq \langle 1_A \rangle \leq (n+1)x$?

The Jiang–Su algebra has the divisibility property of the question above (cf. [15, Lemma 4.2]), and hence so does any unital C^* -algebra that admits a unital embedding of \mathcal{Z} . The question above has an affirmative answer when A is strongly self-absorbing and of stable rank one:

Proposition 5.4 *Let \mathcal{D} be a strongly self-absorbing C^* -algebra of stable rank one such that for each natural number n there is x in the Cuntz semigroup $W(\mathcal{D})$ with $nx \leq \langle 1_{\mathcal{D}} \rangle \leq (n+1)x$. Then the Jiang–Su algebra \mathcal{Z} embeds unittally into \mathcal{D} .*

Proof: One can write \mathcal{Z} as an inductive limit of prime dimension-drop C^* -algebras of the form $Z_{n,n+1}$. By assumption and Proposition 5.1, each $Z_{n,n+1}$ maps unittally into \mathcal{D} . As \mathcal{D} is strongly self-absorbing, \mathcal{D} embeds unittally into $\mathcal{D}_{\infty} \cap \mathcal{D}'$, where $\mathcal{D}_{\infty} = \ell^{\infty}(\mathcal{D})/c_0(\mathcal{D})$, whence $Z_{n,n+1}$ maps

unitally into $\mathcal{D}_\infty \cap \mathcal{D}'$ for all n . It now follows from [16, Proposition 2.2] that $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Z}$, and hence that \mathcal{Z} embeds unitally into \mathcal{D} . \square

We are now ready to prove our main result of this section:

Theorem 5.5 *Let \mathcal{D} be a unital C^* -algebra. Then $\mathcal{D} \cong \mathcal{Z}$ if and only if*

- (i) \mathcal{D} is strongly self-absorbing,
- (ii) the stable rank of \mathcal{D} is one,
- (iii) for all n there is an element $x \in W(\mathcal{D})$ such that $nx \leq \langle 1_{\mathcal{D}} \rangle \leq (n+1)x$,
- (iv) $\mathcal{D} \otimes B \cong B$ for all UHF-algebras B .

Proof: It is well-known that \mathcal{Z} satisfies (i)–(iv). To prove the “if” part it suffices to show that \mathcal{D} embeds unitally into \mathcal{Z} and that \mathcal{Z} embeds unitally into \mathcal{D} , cf. Proposition 2.1.

It follows from Proposition 3.1 that $\mathcal{D} \otimes Z_{2^\infty, 3^\infty} \cong Z_{2^\infty, 3^\infty}$. Hence \mathcal{D} embeds unitally into $Z_{2^\infty, 3^\infty}$ which again embeds unitally into \mathcal{Z} by Proposition 3.3.

That \mathcal{Z} embeds into \mathcal{D} follows from Proposition 5.4. \square

6 Strongly self-absorbing C^* -algebras with almost unperforated Cuntz semigroup

In this section, we rephrase Theorem 5.5 in terms of an algebraic condition on the Cuntz semigroup of a strongly self-absorbing C^* -algebra. Along the way, we show that a strongly self-absorbing C^* -algebra has almost unperforated Cuntz semigroup if and only if it absorbs the Jiang–Su algebra.

Remark 6.1 (Dimension functions) A *dimension function* on a C^* -algebra A is a function $d: M_\infty(A)^+ \rightarrow \mathbb{R}^+$ which satisfies $d(a \oplus b) = d(a) + d(b)$, and $d(a) \leq d(b)$ if $a \precsim b$ for all $a, b \in M_\infty(A)^+$. It is *lower semicontinuous* if, for every monotone increasing sequence (a_n) in $M_\infty(A)^+$ with $a_n \rightarrow a$ for some $a \in A$, one has $d(a_n) \rightarrow d(a)$.

If τ is a (positive) trace on A , then

$$d_\tau(a) = \lim_{n \rightarrow \infty} \tau(a^{1/n}) = \lim_{\varepsilon \rightarrow 0+} \tau(f_\varepsilon(a)), \quad a \in M_\infty(A)^+,$$

defines a dimension function on A (where f_ε is as defined in Notation 4.4, and when τ is extended in the canonical way to $M_\infty(A)$). Every lower semicontinuous dimension function on an exact C^* -algebra arises in this way.

Every dimension function d on A factors through the Cuntz semigroup, i.e., it gives rise to an additive order preserving mapping $\tilde{d}: W(A) \rightarrow \mathbb{R}^+$ given by $\tilde{d}(\langle a \rangle) = d(a)$ for $a \in M_\infty(A)^+$. The functional \tilde{d} is called a *state* (or a *dimension function*) on $W(A)$. If there is no risk of confusion, then we use the same symbol to denote the dimension function on A and the corresponding state on $W(A)$.

It is well-known that a stably finite strongly self-absorbing C^* -algebra \mathcal{D} has precisely one trace (which we shall usually denote by τ); this determines a unique lower semicontinuous dimension function (also denoted by d_τ in the sequel). When identifying \mathcal{D} with $\mathcal{D} \otimes \mathcal{D}$ one has

$$d_\tau(a \otimes b) = d_\tau(a) \cdot d_\tau(b) \tag{6.1}$$

for all $a, b \in \mathcal{D}^+$.

Remark 6.2 (Almost unperforation and strict comparison) The Cuntz semigroup $W(A)$ of a C^* -algebra A is said to be *almost unperforated*, cf. [15], if for all $x, y \in W(A)$ and for all natural numbers n one has $(n+1)x \leq ny \Rightarrow x \leq y$.

If A is simple and unital, then $W(A)$ is almost unperforated if and only if A has *strict comparison*, i.e., whenever $x, y \in W(A)$ are such that $d(x) < d(y)$ for all dimension functions d on A (that can be taken to be normalized: $d(\langle 1_A \rangle) = 1$), then $x \leq y$ (see [15, Proposition 3.2]).

If A is simple, exact and unital, then $W(A)$ is almost unperforated if and only if A has *strict comparison given by traces*: For all $x, y \in W(A)$ one has that $x \leq y$ if $d_\tau(x) < d_\tau(y)$ for all tracial states τ on A , (see [15, Corollary 4.6]).

Lemma 6.3 *Let $A \neq \mathbb{C}$ be a unital C^* -algebra with a faithful tracial state τ . Then there are $0 < \lambda < 1$ and positive elements e and f in A such that $e \perp f$ and*

$$d_\tau(e) = \lambda, \quad d_\tau(f) = 1 - \lambda.$$

Proof: Choose a positive normalized element $d \in A$ such that $\{0, 1\} \subseteq \sigma(d)$; such an element exists in any C^* -algebra of vector space dimension strictly larger than 1. If $\sigma(d) \neq [0, 1]$, then A contains a nontrivial projection p , and we can take $\lambda = \tau(p)$, $e = p$ and $f = 1 - p$. Suppose now that $\sigma(d) = [0, 1]$. The trace τ induces a probability measure μ on $\sigma(d) = [0, 1]$ which is non-zero on any non-empty open subset of $[0, 1]$ (because τ is assumed to be faithful). Take t in the open interval $(0, 1)$ such that $\mu(\{t\}) = 0$. Then $\lambda = \mu([0, t])$, $e = (d - t)_-$, and $f = (d - t)_+$ are as desired. \square

In the lemmas below it is established that the Cuntz semigroup of a strongly self-absorbing C^* -algebra has a rather strong divisibility property.

Lemma 6.4 *Let \mathcal{D} be a strongly self-absorbing C^* -algebra. There are positive elements $b, c \in \mathcal{D}$ such that $\langle b \rangle = \langle c \rangle$, $b \perp c$, and $d_\tau(b) = d_\tau(c) = 1/2$.*

Proof: We can identify \mathcal{D} with $(\mathcal{D}_0)^{\otimes \infty}$, where \mathcal{D}_0 is (isomorphic to) \mathcal{D} . By Lemma 6.3, there are $0 < \lambda < 1$ and positive elements e, f in \mathcal{D} (that we can assume to have norm equal to 1) such that $e \perp f$, $d_\tau(e) = \lambda$, and $d_\tau(f) = 1 - \lambda$. Set $\bar{\lambda} = \lambda(1 - \lambda) > 0$, and set

$$b_0 = e \otimes f \otimes 1_{\mathcal{D}_0} \otimes \cdots \in \mathcal{D}, \quad c_0 = f \otimes e \otimes 1_{\mathcal{D}_0} \otimes \cdots \in \mathcal{D}.$$

Then $b_0 \perp c_0$, and $\langle b_0 \rangle = \langle c_0 \rangle$ because \mathcal{D} is strongly self-absorbing (which implies that there is a sequence (u_n) of unitaries in \mathcal{D} such that $u_n^* b_0 u_n \rightarrow c_0$). Moreover, by (6.1), we have $d_\tau(b_0) = d_\tau(c_0) = \bar{\lambda}$.

Set $d = e \otimes e + f \otimes f \in \mathcal{D}_0 \otimes \mathcal{D}_0$, and for each natural number n set

$$b_n = d \otimes \cdots \otimes d \otimes e \otimes f \otimes 1_{\mathcal{D}_0} \otimes \cdots \in \mathcal{D}, \quad c_n = d \otimes \cdots \otimes d \otimes f \otimes e \otimes 1_{\mathcal{D}_0} \otimes \cdots \in \mathcal{D},$$

where d appears n times. Then, as above, we have that $\langle b_n \rangle = \langle c_n \rangle$; and the elements $b_0, b_1, b_2, \dots, c_0, c_1, c_2, \dots$ are pairwise orthogonal. Moreover, by (6.1), we have $d_\tau(d) = 1 - 2\bar{\lambda}$ and hence

$$d_\tau(b_n) = d_\tau(c_n) = (1 - 2\bar{\lambda})^n \bar{\lambda}.$$

It follows that

$$\sum_{n=0}^{\infty} d_\tau(b_n) = \sum_{n=0}^{\infty} d_\tau(c_n) = 1/2,$$

whence the norm-convergent sums

$$b := \sum_{n=0}^{\infty} \frac{1}{n+1} b_n, \quad c := \sum_{n=0}^{\infty} \frac{1}{n+1} c_n,$$

define elements b and c in \mathcal{D} with the desired properties. \square

Lemma 6.5 *Let A be a C^* -algebra which contains an increasing sequence (A_n) of sub- C^* -algebras whose union is dense in A . Let $x \in W(A)$ and $\varepsilon > 0$ be given. Let τ be a trace on A . Then there exist natural numbers k and r and a positive element $a \in M_r(A_k) \subseteq M_r(A)$ such that $\langle a \rangle \leq x$ and $d_\tau(\langle a \rangle) \geq d_\tau(x) - \varepsilon$.*

Proof: The element x is represented by a positive element b in a matrix algebra $M_r(A)$ over A . Since d_τ is lower semicontinuous there is $\delta > 0$ such that $d_\tau((b - \delta)_+) \geq d_\tau(b) - \varepsilon$. Find k and a positive element a_0 in $M_r(A_k)$ such that $\|a_0 - b\| < \delta/2$. Put $a = (a_0 - \delta/2)_+ \in M_r(A_k)$. Then $\langle a \rangle \leq \langle b \rangle = x$ (by [13, Section 2]). Moreover, $\|a - b\| < \delta$, so again by [13, Section 2], we have $\langle a \rangle \geq \langle (b - \delta)_+ \rangle$, which implies that $d_\tau(\langle a \rangle) \geq d_\tau(\langle (b - \delta)_+ \rangle) \geq d_\tau(x) - \varepsilon$. \square

Lemma 6.6 *Let \mathcal{D} be a strongly self-absorbing C^* -algebra. Let $x \in W(\mathcal{D})$ and $0 \neq k \in \mathbb{N}$ be given. Then, for each $\varepsilon > 0$, there is $y \in W(\mathcal{D})$ such that $ky \leq x$ and $kd_\tau(y) \geq d_\tau(x) - \varepsilon$.*

Proof: Let us first prove the lemma for $k = 2$ (for $k = 1$, there is nothing to show). For each natural number r , identify $M_r(\mathcal{D})$ with $M_r(\mathcal{D}_0) \otimes (\mathcal{D}_0)^{\otimes \infty}$, where \mathcal{D}_0 is (isomorphic to) \mathcal{D} . By Lemma 6.5 it suffices to consider the case where $x = \langle d \rangle$ for some positive element

$$d \in M_r(\mathcal{D}_0) \otimes (\mathcal{D}_0)^{\otimes k} \otimes 1_{\mathcal{D}_0} \otimes \cdots,$$

for suitable natural numbers k and r , that is $d = d_0 \otimes 1_{\mathcal{D}_0} \otimes \cdots$ for some $d_0 \in M_r(\mathcal{D}_0) \otimes (\mathcal{D}_0)^{\otimes k}$. Let b and c be as in Lemma 6.4, and set

$$b' = d_0 \otimes b \in M_r(\mathcal{D}), \quad c' = d_0 \otimes c \in M_r(\mathcal{D}),$$

where we have identified $M_r(\mathcal{D})$ with $M_r(\mathcal{D}_0) \otimes (\mathcal{D}_0)^{\otimes k} \otimes (\mathcal{D}_0)^{\otimes \infty}$. Then b' and c' are orthogonal, belong to the hereditary sub- C^* -algebra of $M_r(\mathcal{D})$ generated by d , and satisfy $\langle b' \rangle = \langle c' \rangle$. Moreover, by (6.1),

$$d_\tau(b') = d_\tau(c') = d_\tau(d)/2.$$

Set $y = \langle b' \rangle$. Then $2y = \langle b' + c' \rangle \leq \langle d \rangle = x$, and $2d_\tau(y) = 2d_\tau(b') = d_\tau(x)$. (Note that in this case, i.e., for $k = 2$ and for $x = \langle d \rangle$ of the special form considered above, we prove the lemma with $\varepsilon = 0$.)

Next, a repeated application of the case $k = 2$ yields that the lemma holds for $k = 2^j$, for any $j \in \mathbb{N}$.

To derive the lemma for an arbitrary natural number k , choose $m, j \in \mathbb{N}$ such that

$$\frac{1}{k} - \frac{\varepsilon}{2kd_\tau(x)} \leq \frac{m}{2^j} \leq \frac{1}{k}.$$

Then

$$2^j \left(1 - \frac{\varepsilon}{2d_\tau(x)}\right) \leq mk \leq 2^j.$$

Choose $\varepsilon_0 > 0$ such that

$$(d_\tau(x) - \varepsilon_0) \left(1 - \frac{\varepsilon}{2d_\tau(x)}\right) \geq d_\tau(x) - \varepsilon.$$

Now apply the lemma with 2^j and ε_0 in the place of k and ε to obtain $y_0 \in W(\mathcal{D})$ with $2^j y_0 \leq x$ and $2^j d_\tau(y_0) \geq d_\tau(x) - \varepsilon_0$. Put $y = my_0$. Then $ky = km y_0 \leq 2^j y_0 \leq x$ and

$$kd_\tau(y) = mk d_\tau(y_0) \geq 2^j \left(1 - \frac{\varepsilon}{2d_\tau(x)}\right) d_\tau(y_0) \geq \left(1 - \frac{\varepsilon}{2d_\tau(x)}\right) (d_\tau(x) - \varepsilon_0) \geq d_\tau(x) - \varepsilon.$$

\square

Proposition 6.7 *Let \mathcal{D} be a strongly self-absorbing C^* -algebra. Then $W(\mathcal{D})$ is almost unperforated if and only if \mathcal{D} absorbs the Jiang–Su algebra tensorially.*

Proof: By [15], \mathcal{Z} -stability implies that the Cuntz semigroup is almost unperforated. To show the converse, it will be enough to consider finite \mathcal{D} , for if \mathcal{D} is infinite, it is well known to absorb \mathcal{O}_∞ , hence \mathcal{Z} . We show that Proposition 5.1(ii) holds for each natural number n , which then, by Proposition 5.1, will imply that $Z_{n,n+1}$ embeds unittally into \mathcal{D} . As in the proof of Proposition 5.4, this entails that \mathcal{Z} embeds unittally into \mathcal{D} . We can finally use Proposition 2.1 to conclude that \mathcal{D} is \mathcal{Z} -stable.

Our proof of 5.1 (ii) follows to a large extent that of (i) \Rightarrow (ii) of Proposition 5.1; however, we will have to avoid use of Lemma 4.5, since we do not assume \mathcal{D} to be of stable rank one.

Let $n \in \mathbb{N}$ be given. By Lemma 6.6 there is $x \in W(\mathcal{D})$ such that $nx \leq \langle 1_{\mathcal{D}} \rangle$ and $d_\tau(x) > 1/(n+1)$. Now follow the proof of (i) \Rightarrow (ii) of Proposition 5.1 to the point where $\delta > 0$, $d \in M_k(\mathcal{D})$, and pairwise orthogonal positive elements e_1, e_2, \dots, e_n in \mathcal{D} have been constructed such that $x = \langle d \rangle$ and $e_j \sim (d - \delta)_+$. (Note that the assumption of stable rank one was not used up to that point.) Upon choosing $\delta > 0$ small enough, and using lower semicontinuity of d_τ , one can further obtain that $d_\tau(e_1) = d_\tau((d - \delta)_+) > 1/(n+1)$ (recalling that $d_\tau(d) = d_\tau(x) > 1/(n+1)$). For $\eta > 0$, let f_η be as in Notation 4.4. As $1_{\mathcal{D}} - f_\eta(e_1 + e_2 + \dots + e_n) \perp (e_1 + e_2 + \dots + e_n - \eta)_+$ we get

$$\begin{aligned} \lim_{\eta \rightarrow 0+} d_\tau(1_{\mathcal{D}} - f_\eta(e_1 + e_2 + \dots + e_n)) &\leq 1 - \lim_{\eta \rightarrow 0+} d_\tau((e_1 + e_2 + \dots + e_n - \eta)_+) \\ &= 1 - nd_\tau(e_1) \\ &< d_\tau(e_1) \\ &= \lim_{\eta \rightarrow 0+} d_\tau((e_1 - \eta)_+) \end{aligned}$$

(the first inequality is actually equality). Thus we infer, by Remark 6.2 and the assumption that $W(\mathcal{D})$ is almost unperforated, that

$$1_{\mathcal{D}} - f_\eta(e_1 + e_2 + \dots + e_n) \precsim (e_1 - \eta)_+$$

for some $\eta > 0$. We can now follow the last three lines of the proof of (i) \Rightarrow (ii) of Proposition 5.1 to arrive at the conclusion that 5.1(ii) holds. \square

Corollary 6.8 *In Theorem 5.5, conditions (ii) and (iii) may as well be replaced by*

(ii') \mathcal{D} is finite

(iii') $W(\mathcal{D})$ is almost unperforated.

Proof: By [8] and [15], \mathcal{Z} satisfies (ii') and (iii'), so we have to check that (ii') and (iii') (together with the other hypotheses) imply conditions (ii) and (iii) of 5.5. But (iii') entails that \mathcal{D} is \mathcal{Z} -stable by Proposition 6.7, and (ii') together with \mathcal{Z} -stability yields stable rank one, cf. [15]. Now by Remark 5.2, \mathcal{D} satisfies 5.1(i), hence 5.5(iii). \square

7 Strongly self-absorbing C^* -algebras with finite decomposition rank

In this final section we single out the Jiang–Su algebra among strongly self-absorbing C^* -algebras with finite decomposition rank. Recall that the latter is a notion of topological dimension for nuclear C^* -algebras that was introduced by E. Kirchberg and the second named author in [11].

The order on the Cuntz semigroup is not the algebraic order (i.e., if $x \leq y$, then we do not necessarily have z in the Cuntz semigroup such that $y = x + z$). The following lemma, which is needed for the proof of Proposition 7.5 below, seeks to remedy this situation.

Lemma 7.1 *Let A be a C^* -algebra.*

- (i) *Let a, b be positive elements in A such that $a \precsim b$, and let $\varepsilon > 0$ be given. Then there are positive elements a_0 and c in \overline{bAb} such that*

$$a_0 \perp c, \quad a_0 \sim (a - 2\varepsilon)_+, \quad b \precsim (a - \varepsilon)_+ \oplus c.$$

Moreover, if d is a lower semicontinuous dimension function on A and if $\delta > 0$ is given, then there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$, then

$$d(b) - d(a) \leq d(c) \leq d(b) - d(a) + \delta.$$

- (ii) *Let d be a lower semicontinuous dimension function on $W(A)$, and let $x, y \in W(A)$ be such that $x \leq y$. Then, for each $\delta > 0$, there is $z \in W(A)$ such that $x + z \geq y$ and $d(z) \leq d(y) - d(x) + \delta$.*

Proof: (i). By [13, Proposition 2.4] there is $v \in A$ such that $v^*v = (a - \varepsilon)_+$ and vv^* belongs to \overline{bAb} . With h_ε as defined in (4.1) we have $h_\varepsilon(vv^*) \perp (vv^* - \varepsilon)_+$. (We remark that $h_\varepsilon(vv^*)$ belongs to A if A is unital, and that it otherwise belongs to the unitization of A .) Put

$$a_0 = (vv^* - \varepsilon)_+ \sim (v^*v - \varepsilon)_+ = (a - 2\varepsilon)_+, \quad c = h_\varepsilon(vv^*)bh_\varepsilon(vv^*),$$

and note that a_0 and c both belong to \overline{bAb} . Moreover, $a_0 \perp c$, and $vv^* + c$ is strictly positive in \overline{bAb} . The latter implies that

$$b \precsim vv^* + c \precsim vv^* \oplus c \sim (a - \varepsilon)_+ \oplus c.$$

If d is a lower semicontinuous dimension function on A , then for each $\delta > 0$ there is $\varepsilon_0 > 0$ such that $d((a - 2\varepsilon_0)_+) \geq d(a) - \delta$. As $a_0 \perp c$ we have $d(a_0) + d(c) = d(a_0 + c) \leq d(b)$, whence

$$d(c) \leq d(b) - d(a_0) = d(b) - d((a - 2\varepsilon)_+) \leq d(b) - d((a - 2\varepsilon_0)_+) \leq d(b) - d(a) + \delta,$$

whenever $0 < \varepsilon \leq \varepsilon_0$. On the other hand, since $b \precsim (a - \varepsilon)_+ \oplus c$ we have $d(b) \leq d((a - \varepsilon)_+) + d(c) \leq d(a) + d(c)$, which entails that $d(c) \geq d(b) - d(a)$.

(ii). Upon replacing A with a matrix algebra over A we can assume that $x = \langle a \rangle$ and $y = \langle b \rangle$ for some positive elements $a, b \in A$. Now use (i) to find $\varepsilon > 0$ and c such that $b \precsim (a - \varepsilon)_+ \oplus c \precsim a \oplus c$ and such that $d(c) \leq d(b) - d(a) + \delta$. We can then take z to be $\langle c \rangle$. \square

We quote below a result by Andrew Toms and the second named author stating that C^* -algebras with finite decomposition rank satisfy a weak version of strict comparison. The original lemma was stated in a slightly different manner; the version below employs the fact that decomposition rank is invariant under taking matrix algebras.

Lemma 7.2 (Toms–Winter, [17, Lemma 6.1]) *Let A be a simple, separable and unital C^* -algebra with decomposition rank $n < \infty$. Suppose that $x, y_0, y_1, \dots, y_n \in W(A)$ satisfy $d(x) < d(y_j)$ for all $j = 0, 1, \dots, n$ and for any lower semicontinuous dimension function d on A . Then $x \leq y_0 + y_1 + \dots + y_n$.*

The lemma above has the following two sharper versions for strongly self-absorbing C^* -algebras:

Lemma 7.3 *Let \mathcal{D} be strongly self-absorbing with decomposition rank $n < \infty$, and let $x, y \in W(\mathcal{D})$ with $(n + 1)d_\tau(x) < d_\tau(y)$ be given. Then $x \leq y$.*

Proof: Apply Lemma 6.6 with $k = n + 1$ to obtain $z \in W(\mathcal{D})$ such that $(n + 1)z \leq y$ and

$$(n + 1)d_\tau(z) > d_\tau(y) - (d_\tau(y) - (n + 1)d_\tau(x)) = (n + 1)d_\tau(x);$$

we then have $d_\tau(x) < d_\tau(z)$. Now from Lemma 7.2 we obtain $x \leq (n + 1)z \leq y$. \square

Lemma 7.4 *Let \mathcal{D} be strongly self-absorbing with decomposition rank $n < \infty$, and let $x, y, z \in W(\mathcal{D})$ be such that $x \leq y$ and $(n+1)d_\tau(z) < d_\tau(y) - d_\tau(x)$. Then $x + z \leq y$.*

Proof: We may assume that $x = \langle a \rangle$, $y = \langle b \rangle$ and $z = \langle e \rangle$, where a, b and e are positive elements in some matrix algebra $M_r(\mathcal{D})$ over \mathcal{D} . To show that $x + z \leq y$ it suffices to show that $(a - 2\varepsilon)_+ \oplus e \preceq b$ for all $\varepsilon > 0$.

By Lemma 7.1 (i) there are mutually orthogonal positive elements a_0 and c in the hereditary sub- C^* -algebra of $M_r(\mathcal{D})$ generated by b such that $a_0 \sim (a - 2\varepsilon)_+$ and $d_\tau(c) \geq d_\tau(b) - d_\tau(a) > (n+1)d_\tau(e)$. But then it follows from Lemma 7.3 that $e \preceq c$, whence

$$(a - 2\varepsilon)_+ \oplus e \preceq (a - 2\varepsilon)_+ \oplus c \sim a_0 \oplus c \sim a_0 + c \preceq b,$$

as desired. \square

Proposition 7.5 *Any strongly self-absorbing C^* -algebra \mathcal{D} with finite decomposition rank absorbs the Jiang–Su algebra \mathcal{Z} , i.e., $\mathcal{D} \otimes \mathcal{Z} \cong \mathcal{D}$.*

Proof: By Remark 6.2 and Proposition 6.7 it suffices to show that for all $x, y \in W(\mathcal{D})$ with $d_\tau(x) < d_\tau(y)$ one has $x \leq y$, where τ is the unique trace on \mathcal{D} . Put $\delta = (d_\tau(y) - d_\tau(x))/(n+1)$, where n is the decomposition rank of \mathcal{D} . Choose an integer $k \geq n$ such that $(n+1)d_\tau(x)/k < \delta$. By Lemma 6.6 there is $x_0 \in W(\mathcal{D})$ such that $kx_0 \leq x$ and $kd_\tau(x_0) \geq d_\tau(x) - \delta/2$, and by Lemma 7.1 (ii) there is $z \in W(\mathcal{D})$ such that $kx_0 + z \geq x$ and $d_\tau(z) < d_\tau(x) - kd_\tau(x_0) + \delta/2 \leq \delta$. For each $j = 0, 1, \dots, n-1$ we have

$$(n+1)d_\tau(x_0) \leq (n+1)d_\tau(x)/k < \delta = d_\tau(y) - d_\tau(x) \leq d_\tau(y) - d_\tau(jx_0).$$

Lemma 7.4 therefore yields $jx_0 \leq y \Rightarrow (j+1)x_0 \leq y$ for $j = 0, 1, \dots, n-1$. Hence $nx_0 \leq y$. Next,

$$(n+1)d_\tau(z) < (n+1)\delta \leq d_\tau(y) - d_\tau(x) \leq d_\tau(y) - d_\tau(nx_0),$$

so, again by Lemma 7.4, we get $x \leq nx_0 + z \leq y$ as desired. \square

Theorem 7.6 *Let \mathcal{D} be a unital C^* -algebra. Then $\mathcal{D} \cong \mathcal{Z}$ if and only if*

- (i) \mathcal{D} is strongly self-absorbing,
- (ii) the decomposition rank of \mathcal{D} is finite,
- (iii) \mathcal{D} is KK -equivalent to \mathbb{C} .

Proof: It is well-known that \mathcal{Z} satisfies properties (i)–(iii) above. Assume now that (i)–(iii) holds. To show that $\mathcal{D} \otimes \mathcal{Z} \cong \mathcal{Z}$, note that $\mathcal{D} \otimes \mathcal{Z}$ and \mathcal{Z} both have (locally) finite decomposition rank, and are \mathcal{Z} -stable. Since $\mathcal{D} \otimes \mathcal{Z}$ is KK -equivalent to \mathbb{C} , $K_0(\mathcal{D} \otimes \mathcal{Z}) = \mathbb{Z}$ and $K_1(\mathcal{D} \otimes \mathcal{Z}) = 0$. Since $\mathcal{D} \otimes \mathcal{Z}$ is stably finite and \mathcal{Z} -stable, the order structure of its K -theory is determined by the unique tracial state (see [6]), whence $\mathcal{D} \otimes \mathcal{Z} \cong \mathcal{Z}$ by [22, Corollary 8.1].

That $\mathcal{D} \otimes \mathcal{Z} \cong \mathcal{D}$ simply follows from Proposition 7.5. \square

Remarks 7.7 Formally, Theorems 5.5 and 7.6 are very similar, and it is interesting to compare them. Conditions (ii) of both theorems refer to notions of noncommutative covering dimension; however, one should keep in mind that decomposition rank has a much more topological flavor than stable rank one.

Furthermore, using [7] and the fact that (generalized) prime dimension drop C^* -algebras are KK -equivalent to \mathbb{C} (cf. [8]), 7.6(iii) follows from 5.5(iv). Because of these conditions, neither 5.5 nor 7.6 are completely intrinsic characterizations

Condition 5.5(iii) may be interpreted as a K -theory type condition in the broadest sense; it remains an interesting possibility that it is redundant in 5.5. Similarly, it might be the case that 7.6 still holds when only asking for locally finite (as opposed to finite) decomposition rank in

7.6(ii). Conditions 5.5(iii) and 7.6(ii) are (implicitly) both used to ensure notions of comparison of positive elements. So, the question is whether (stably finite) strongly self-absorbing C^* -algebras automatically have some sort of comparison property. (In the infinite case, this has an affirmative answer, since an infinite strongly self-absorbing C^* -algebra is always purely infinite by a result of Kirchberg.)

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